Name: ____________________ Date: ____________

Rational Exponents and Radicals

Objective

In this lesson, you will rewrite radical expressions and expressions with rational exponents.

Rewriting Radical Expressions

Examine the radical expression $\sqrt[3]{x^2}$. It can be converted to an expression with rational **exponents** by applying the following property: $\sqrt[n]{\boldsymbol{x^m}} = \boldsymbol{x^{\frac{m}{n}}}$.

In this case, *m* = **2** and *n* = **3**. Write the exponential expression where the base of the exponent from the radicand, *x*, is raised to the power of 2 **over** 3. So, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

To work in the reverse direction, start with an exponential expression $y^{\frac{9}{2}}$. Here, *m* = **9** and *n* = **2**. Write the radical expression with the base of the exponential expression, *y*, raised to the power of **9** under a radical with an **index** of 2. So, $y^{\frac{9}{2}} = \sqrt{y^9}$.

Question Match each radical expression or exponential expression with its equivalent expression of the opposite form:

$\sqrt[3]{b^5}$	$b^{\frac{5}{3}}$
$\sqrt[5]{b^3}$	$b^{\frac{3}{5}}$
$b^{\frac{5}{2}}$	$\sqrt{b^5}$
$b^{\frac{2}{5}}$	$\sqrt[5]{b^2}$

PROPERTIES OF RATIONAL EXPONENTS

Product of Powers Property: $a^m \cdot a^n = a^{m+n}$ **Addition** of fractions works the same way as integers. Apply this property when the powers are fractions.	$5^{\frac{1}{2}} \cdot 5^{\frac{1}{4}} = 5^{\frac{1}{2}+\frac{1}{4}}$ $= 5^{\frac{2}{4}+\frac{1}{4}}$ $= 5^{\frac{3}{4}}$
Power of a Power Property: $(a^m)^n = a^{mn}$ **Multiplication** of fractions works the same way as integers. Apply this property when the powers are fractions.	$\left(3^{\frac{1}{2}}\right)^2 = 3^{\frac{1}{2}\cdot 2}$ $= 3^1$ $= 3$
Power of a Product Property: $(ab)^m = a^m b^m$ Raising a product of numbers to a rational exponent works the same as raising a product to an integer power. Apply the rational exponent to each number inside the parentheses **separately**.	$(9x)^{\frac{1}{2}} = \boxed{9}^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$
Negative Exponent Property: $a^{-m} = \frac{1}{a^m}, a \neq 0$ A negative rational exponent works the same way as a negative integer exponent. Write the expression as a fraction, with **1** as the numerator and the denominator as the positive of the original rational exponent.	$27^{-\frac{1}{3}} = \frac{1}{\boxed{27}^{\frac{1}{3}}}$
Quotient of Powers Property: $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$ **Subtraction** of fractions works the same way as integers. Apply this property when the exponents are fractions.	$\frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = x^{\frac{2}{3}-\frac{1}{3}}$ $= x^{\frac{\boxed{1}}{3}}$
Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$ Raising a quotient of numbers to a fraction works the same as raising a quotient to an integer. We apply the rational **exponent** to both the numerator and denominator **separately**.	$\left(\frac{y}{8}\right)^{\frac{2}{3}} = \frac{y^{\frac{2}{3}}}{8^{\frac{2}{3}}}$

The process of simplifying a numerical radical expression involves rewriting it so the radicand is the smallest possible whole number. This process may involve factorizing the radicand or using rational exponents to rewrite the expression.

Using Factorization	Using Rational Exponents
$\sqrt{125} = \sqrt{5 \cdot 5 \cdot 5}$ $= \boxed{5}\sqrt{5}$	$\sqrt{125} = \sqrt{5^2 \cdot 5}$ $= (5^2)^{\frac{1}{\boxed{2}}}\sqrt{5}$ $= 5\sqrt{5}$
$\sqrt[3]{125g^4h^3} = \sqrt[3]{5 \cdot 5 \cdot 5 \cdot g \cdot g \cdot g \cdot g \cdot h \cdot h \cdot h}$ $= \boxed{5} \cdot g \cdot h \cdot \sqrt[3]{\boxed{g}}$ $= 5gh\sqrt[3]{g}$	$\sqrt[3]{125g^4h^3} = \sqrt[3]{5^3 \cdot g^3 \cdot g \cdot h^3}$ $= \left(5^{\boxed{3}}\right)^{\frac{1}{3}} \cdot (g^3)^{\frac{1}{3}} \cdot (h^3)^{\frac{1}{3}} \cdot \sqrt[3]{g}$ $= 5 \cdot g \cdot h \cdot \sqrt[3]{g}$ $= 5gh\sqrt[3]{g}$

Example: Rewrite the expression $\sqrt{54x^5y^8}$ with rational exponents and use the properties of exponents to simplify the expression.

1. This is a square root expression, so rewrite the radicand using **perfect square** factors. Then use rational exponents to write the perfect square factors in the **radicand** as exponential expressions.

$$\sqrt{54x^5y^8} = \sqrt{3^2 \cdot 6 \cdot x^2 \cdot x^2 \cdot \boxed{x} \cdot y^2 \cdot y^2 \cdot y^2 \cdot y^2}$$
$$= (3^2)^{\frac{1}{2}} \cdot (x^2)^{\frac{1}{2}} \cdot (x^2)^{\frac{1}{2}} \cdot (y^2)^{\frac{1}{2}} \cdot (y^2)^{\frac{1}{2}} \cdot (y^2)^{\frac{1}{2}} \cdot (y^2)^{\frac{1}{2}} \cdot \sqrt{\boxed{6} \cdot x}$$

2. Use the power of a **power** property to simplify the exponential expressions.

$$= \boxed{3} \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot \sqrt{6 \cdot x}$$

3. Multiply the expressions **outside** the radical to get the final, simplified version of the original expression.

$$= 3x^{\boxed{2}}y^{\boxed{4}}\sqrt{6x}$$

Performing Operations

Radical expressions can consist of more than one term.

ADDITION AND SUBTRACTION

Simplify expressions by combining like terms. Like terms must have the same **variables** raised to the same **power**, and if the terms include radicals, they must be like terms as well. When radicals are like terms, they have the same **index** and **radicand**. To determine whether like terms are present, we need to simplify each term and examine the result.

Example: Examine the expression $2\sqrt[3]{6x} - \sqrt[3]{6x} + 4\sqrt[3]{6x}$ to determine whether it has like terms.

- All three radicals are **cube** roots of the same expression, **6x**, so they are like terms.
- Use the **distributive** property to rewrite the given expression.
- Simplify the expression **inside** the parentheses.

$$2\sqrt[3]{6x} - \sqrt[3]{6x} + 4\sqrt[3]{6x} = (2 - 1 + 4)\sqrt[3]{6x}$$
$$= 5\sqrt[3]{6x}$$

Example: Use simplifying methods to rewrite the expression $3\sqrt{8x^3} + x\sqrt{50x}$.

Rewrite the terms of the expression using **rational** exponents. Use the properties of exponents to simplify the terms.	$3\sqrt{8x^3} + x\sqrt{50x} = 3\sqrt{2^2 \cdot \boxed{2} \cdot x^2 \cdot x} + x\sqrt{5^2 \cdot \boxed{2} \cdot x}$ $= 3 \cdot (2^2)^{\frac{1}{2}} \cdot (x^2)^{\frac{1}{2}} \cdot \sqrt{2 \cdot x} + x \cdot (5^2)^{\frac{1}{2}} \cdot \sqrt{2 \cdot x}$ $= 3 \cdot \boxed{2} \cdot x \cdot \sqrt{2 \cdot x} + x \cdot \boxed{5} \cdot \sqrt{2 \cdot x}$ $= \boxed{6}\, x\sqrt{2x} + 5x\sqrt{2x}$ $= (6x + 5x)\sqrt{\boxed{2}\, x}$ $= \boxed{11}\, x\sqrt{2x}$
Once the terms are simplified, both terms are **square** root expressions with the same radicand, **2x**. Add the like terms to find the sum.	

MULTIPLICATION AND DIVISION

Similar to when we perform operations with polynomials, like terms **aren't** required for multiplication or division.

Example: Simplify this product of two radical expressions: $5\sqrt[3]{2y} \cdot 6\sqrt[3]{xy}$.

- Any values **outside** the radicals can be multiplied. For the radicands, use: $\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y}$
- Because the **indices** of the radicals are the same, write one radicand as the product of the two.
- Multiply the values outside and inside the radical.

$$5\sqrt[3]{2y} \cdot 6\sqrt[3]{xy} = 5 \cdot 6 \cdot \sqrt[3]{2y \cdot xy}$$
$$= 30\sqrt[3]{2xy^2}$$

Example: Simplify this product of two radical expressions: $4\sqrt{x} \cdot 2\sqrt[3]{x^2}$.

Rewrite the radicals using rational exponents.	$4\sqrt{x} \cdot 2\sqrt[3]{x^2} = 4x^{\frac{1}{2}} \cdot 2x^{\frac{2}{3}}$
Multiply the coefficients and the exponential expressions using the properties of exponents. When multiplying two powers with the same base, **add** the exponents.	$= 4 \cdot \boxed{2} \cdot x^{\frac{1}{2}} \cdot x^{\frac{2}{3}}$ $= \boxed{8} \cdot x^{\frac{1}{2}+\frac{2}{3}}$ $= 8 \cdot x^{\frac{3}{6}+\frac{4}{6}}$ $= 8 \cdot x^{\frac{\boxed{7}}{6}}$
Rewrite the exponential expression in **radical** form and use the same methods to simplify the expression. This method of multiplication also works for products with the **same** roots, such as the previous example.	$= 8 \cdot \sqrt[6]{x^7}$ $= 8 \cdot (x^6)^{\frac{1}{6}} \cdot \sqrt[6]{\boxed{x}}$ $= \boxed{8}\, x\sqrt[6]{x}$

To divide radical expressions, begin by rewriting the division as a **fraction**.

Use the rule $\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$ to simplify the quotient of the radical parts.

Recall that when dividing powers with the same base, **subtract** the exponents.

Example: Divide the radical expressions with different roots: $\sqrt{36c^3} \div \sqrt[3]{27c}$.

To begin, rewrite the division as a **fraction**.	$\sqrt{36c^3} \div \sqrt[3]{27c} = \dfrac{\sqrt{36c^3}}{\sqrt[3]{27c}}.$
Simplify the expressions in the numerator and the denominator of the fraction using **factorization**.	$= \dfrac{\sqrt{6 \cdot 6 \cdot \boxed{c} \cdot c \cdot c}}{\sqrt[3]{3 \cdot 3 \cdot 3 \cdot c}}$
Separate the values **outside** the radicals from the **radicands**.	$= \dfrac{\boxed{6}\, c\sqrt{c}}{3\sqrt[3]{c}}$ $= \dfrac{6c}{3} \cdot \dfrac{\sqrt{c}}{\sqrt[3]{c}}$
After simplifying the first fraction, divide the radicals in the second fraction by rewriting the radicals in **exponential** form with rational exponents.	$= \boxed{2}\, c \cdot \dfrac{c^{\frac{1}{2}}}{c^{\frac{1}{3}}}$
Divide the exponential expressions using the properties of exponents, and rewrite the simplified exponential expression in **radical** form.	$= 2c \cdot c^{\frac{1}{2}-\frac{1}{3}}$ $= 2c \cdot c^{\frac{3}{6}-\frac{2}{6}}$ $= 2c \cdot c^{\boxed{\frac{1}{6}}}$ $= 2c\sqrt[6]{c}$

RATIONALIZING DENOMINATORS

If dividing radical expressions results in a **radical** in the denominator, the resulting expression isn't considered completely simplified. It can be manipulated and rewritten to eliminate the issue. This is called **rationalizing** the denominator.

$$\frac{\sqrt{y}}{\sqrt{2y^2}} = \sqrt{\frac{y}{2y^2}} = \sqrt{\frac{1}{2y}} = \frac{\sqrt{1}}{\sqrt{2y}} = \frac{1}{\sqrt{2y}}$$

Example: Divide the radical expressions: $\frac{\sqrt{y}}{\sqrt{2y^2}}$ as started above.

→ Multiply the numerator and the denominator by the radical expression in the **denominator**. This is equivalent to multiplying the entire fraction by **1**. Simplify the numerator and denominator.

$$\frac{1}{\sqrt{2y}} \cdot \left(\frac{\sqrt{\boxed{2y}}}{\sqrt{2y}}\right) = \frac{\sqrt{2y}}{\left(\sqrt{2y}\right)^2}$$

$$= \frac{\sqrt{2y}}{\boxed{2y}}$$