Name: ____________________ Date: ______________

Factors, Zeros, and Solutions of Polynomial Equations

Objective

In this lesson, you will investigate how the solutions of a polynomial equation are related to the graph of the corresponding function.

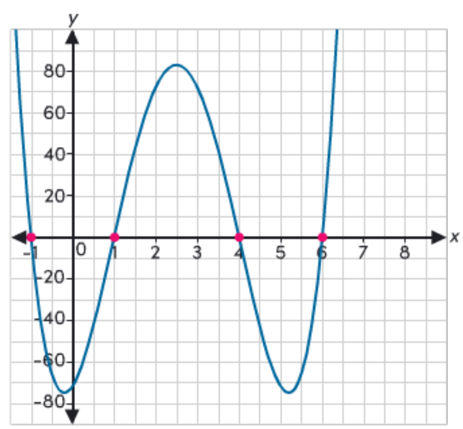
> Zero: the **solution** to an equation; where the graph of a function crosses the ***x*** - **axis**

Polynomials in Factored Form

If **(*x* – *m*)** is a factor of the quadratic, then ***m*** is a zero. Likewise, if *m* is a **zero** of a function, then we can say that (*x* – *m*) is a **factor** of the function.

Let's consider a polynomial with **zeros** -1, 1, 4, and 6.

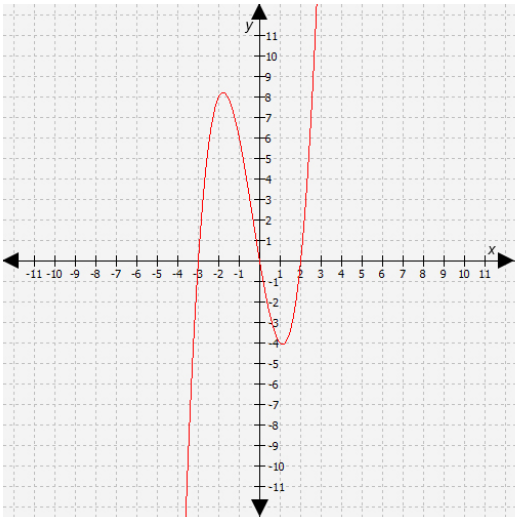
factors	*x*-intercepts
(x + **1**), (*x* – 1), (*x* – **4**), (*x* – 6)	(-1,0), (**1**,0), (**4**,0), (6,0)
The polynomial described could be $f(x) = (x + \boxed{1})(x-1)(x-4)(x - \boxed{6})$	



We can't say for certain that this is the exact polynomial described because we don't know if there are other **factors** of the polynomial. For example, $g(x) = \boxed{3}\,(x+1)(x-1)(x-4)(x-6)$ has the **same** zeros as the described polynomial.

FACTORED FORM FROM A GRAPH

Recall that the *x*-intercepts of the function are also the zeros of the function.



Consider this graph of a cubic function. The graph crosses the x-axis at the points (-3,0), (0,0), and (2,0).

***x*-Intercepts**	(-3,0), (0,0), and (2,0)
Zeros	-3, 0, and 2
Factors	(*x* + 3), *x*, and (*x* – 2)
Equation	*y* = x(*x* + 3)(*x* – 2)

Solutions of Polynomials

- ✓ **Zero product property:** the property stating that if a product is 0, then one or more of the factors must be equal to 0: if *a* • *b* = 0, then *a* = 0, *b* = 0, or both
- ✓ **Degree (of a polynomial):** the largest exponent of a variable in a term of an expression

The solutions, or zeros, of a polynomial function correspond to the *x*-intercepts of the graph of the function.

Find the factors of the function.	Use the zero-product property to find the zeros.
$p(x) = (x^2 - 7x + 12)(x - 1)$ $= (x - 3)(x - 4)(x - 1)$	see table below

x – 4	*x* – 3	*x* – 1
x – 4 = 0 *x* = 4	*x* – 3 = 0 *x* = 3	*x* – 1 = 0 *x* = 1

x-intercepts: (1,0), (3,0), (4,0)

The fundamental theorem of algebra: any polynomial function of degree *n* has *n* zeros.

→ Example: $f(x) = x^4 - 3x^3 + 2$ has four zeros because it has a degree of 4.

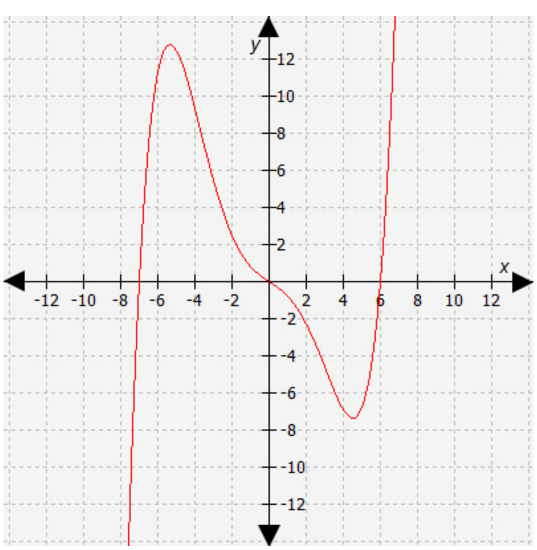
We need to examine the graph, factor, and divide out factors to determine whether the solutions are real or complex.

➢ **Factor:** Identifying Real and Complex Roots

Expand the function to find its **degree**. Function f is a **fourth**-degree polynomial, so it will have **four** zeros.	$f(x) = (x^2 - 3x + 2)(x^2 + 1)$ $= x^4 - \boxed{3}x^3 + 2x^2 + x^2 - 3x + \boxed{2}$ $= x^4 - 3x^3 + \boxed{3}x^2 - 3x + 2$
Use the factored form of function f to find its zeros.	$f(x) = (x^2 - 3x + 2)(x^2 + 1)$ $= (x - \boxed{2})(x - 1)(x^2 + \boxed{1})$
Use the zero product property to find the zeros. Function f has: → two real zeros, **1** and **2**, → two complex zeros, **i** and **-i**	(see table below)
If a function has **complex** zeros, it will contain a term that can't be factored into linear terms. In this case, $x^2 + 1$.	

$x - 2$	$x - 1$	$x^2 + 1$
$x - 2 = \boxed{0}$ $x = 2$	$x - 1 = 0$ $x = \boxed{1}$	$x^2 + 1 = 0$ $x^2 =$ $x = \pm\sqrt{-1}$ $= \pm i$

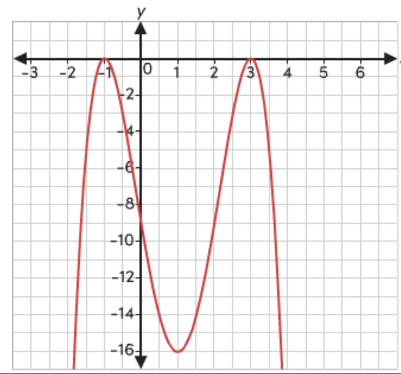
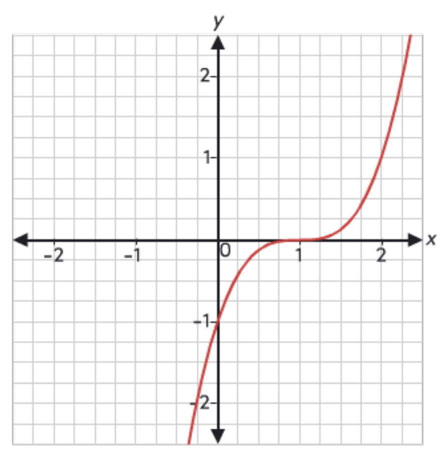
Question



- Because the polynomial has a **degree** of 5, it must have five **zeros**.
- The function crosses the x-axis at -7,0, and 6, which means the function has **three** real zeros.
- By subtracting the number of real zeros from the total number of zeros (5 – 3), we can determine that the function has **two** complex zeros.

When a polynomial function has <u>**repeated**</u>________ factors, such as $(x + 1)^2$, we plot a <u>**single**</u>________ point on the *x*-axis for the repeated zero.

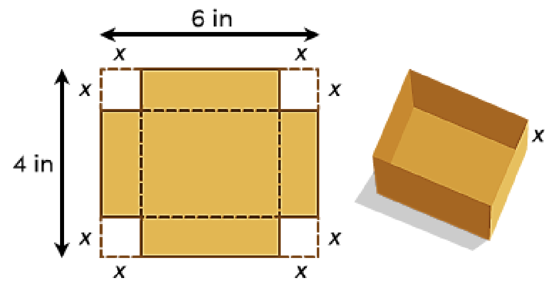
- **Multiplicity**: the number of times a <u>**zero**</u>______ appears in a factored polynomial

Multiplicity	Factor the Equation	Graph Behavior
Touches the graph/ returns: <u>**Even**</u> **Multiplicity**	$f(x) = -(x^2 - 6x + 9)(x^2 + 2x + 1)$ $= -(x-3)(x-3)(x+1)(x+1)$ $= -(x-3)^{\boxed{2}}(x+1)^{\boxed{2}}$	
	The factors reveal zeros at $x =$ <u>**3**</u> and $x =$ <u>**-1**</u>. The multiplicity of both zeros is <u>**2**</u>, so it will "<u>**bounce**</u>" at both these points.	
Passes through the point/ little curve: <u>**Odd**</u> **Multiplicity**	$f(x) = x^3 - 3x^2 + 3x - 1$ $= (x^3 - 1) + (-3x^2 + 3x)$ $= (x-1)(x^2 + x + 1) - \boxed{3}x(x-1)$ $= (x - \boxed{1})(x^2 + x + 1 - 3x)$ $= (x-1)(x^2 - 2x + 1)$ $= (x-1)^{\boxed{3}}$	
	The factors reveal a single zero at $x =$ <u>**1**</u>. The multiplicity of this zero is <u>**1**</u>, so it will "<u>**swoop**</u>" at this point.	

<u>**Complex**</u>________ zeros can't be shown on the *x*-axis. So, we see the graph passing through only the real zeros on the *x*-axis.

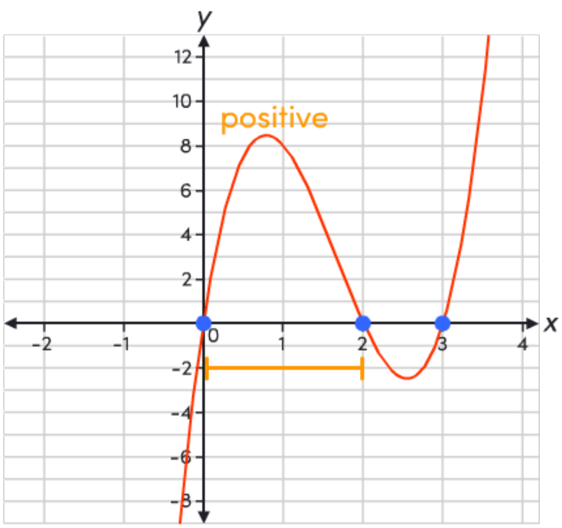
Applications

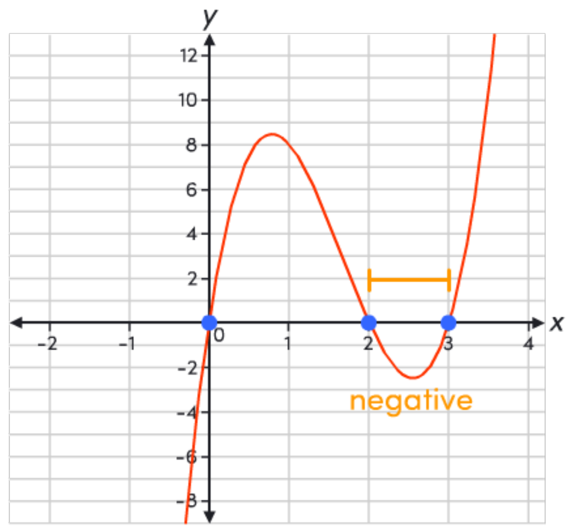
Example 1: Clarissa is making a box out of a 4-inch by 6-inch piece of poster board. She will cut a square with a width of *x* inches from each corner of the poster board and then fold up the edges to make the box. Function *f* models the volume of the box in terms of *x*.



$$f(x) = 4x(x-3)(x-2)$$

The function has **three** real zeros. At these *x*-values, the volume of the box will be **0**, so a box can't be made when *x* = 0, *x* = 2, or *x* = 3.





Between *x* = **0** and *x* = **2**, the function is **positive**. Clarissa can use *x*-values within this range.

Between *x* = **2** and *x* = **3**, the function is **negative**. Since volume can't be negative, Clarissa can't use values between 2 and 3.

The function is positive for values of *x* greater than 4. Are these *x*-values **reasonable solutions**?

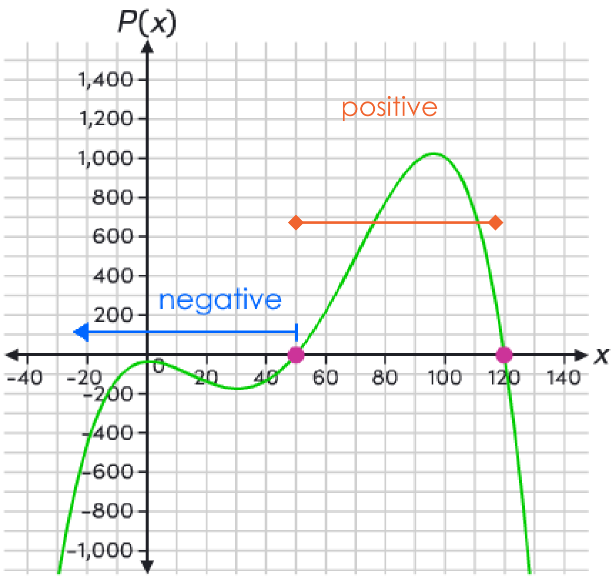
- The width of the poster board is 4 inches. Clarissa can't cut two squares with a width of 4 inches or **greater** out of the poster board.
- So values greater than 4 are **not reasonable** in the real-world context.

Example 2: A company uses function *P* to model its weekly profit, *P*(*x*), for selling *x* units. For what values of x will the company make a profit?

$$P(x) = -0.0001(x^2 + 60)(x - 50)(x - 120)$$

Function *P* has **two real zeros**, *x* = 50 and *x* = 120, and **two complex zeros**.

The company will make a profit of exactly $0, or break even, if it sells 50 units or 120 units.



→ The company does not make a profit when it sells **fewer** than 50 units.

→ The company makes a profit when it sells between **50** and **120** units.

→ The value of *P*(*x*) decreases to a **negative** value beyond *x* = 120. So the company does not make a profit for selling more than **120** units.